The “Salami” Theorem and the search for extraterrestrial life or How to share volume, light and wavefront or *How I can still annoy my friend Bob with mathematical puzzles*

D. Pelat

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Collaborators

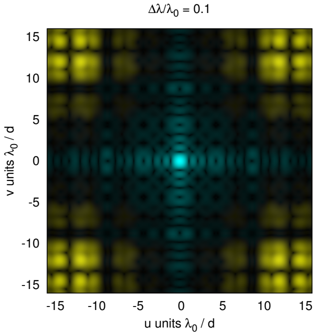
Daniel Rouan, Didier Pelat,

Jean-Michel Reess, Fanny Chemla, Matthieu Cohen, Olivier Dupuis,

Marie Ygouf, Nicolas Meilard, Damien Pickel

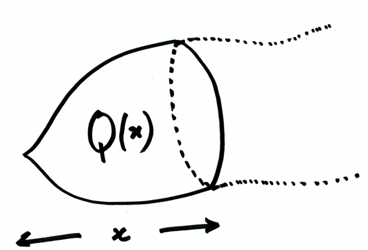
$$P_0 = [0], \quad Q_0 = [1];$$

$$P_{n+1} = \begin{bmatrix} Q_n + 1 & P_n + 2 \\ P_n & Q_n + 1 \end{bmatrix}, \quad Q_{n+1} = \begin{bmatrix} P_n + 1 & Q_n + 2 \\ Q_n & P_n + 1 \end{bmatrix}.$$



How to equitably share a volume, e.g. a "salami"?

Suppose you are given a salami and a knife and have to distribute an equal share of this Italian speciality between two persons, say A and B.
The salami is indeed irregular, but you know that the quantity $Q(x)$ cut at abscissa x is a polynomial[1] $P_n(x)$ of degree n.

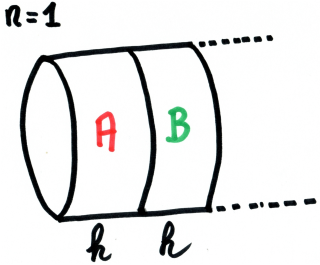


The only information about the polynomial is n its degree, you don't know the polynomial coefficients.

[1] Note that because of the Taylor expansion, many quantities can be accurately approximated by polynomials.

Solution by induction, order 1

First observation: $Q(x)$ is an increasing function of x (there is nothing like a negative salami) i.e. $n \geq 1$.
If $n = 1$, the solution is: cut two slices of equal thickness h; give one to A, say the first one, and the second to B.

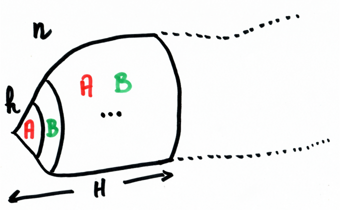


In fact $Q'(x) = c_0 =$ constant and

$$Q_1(A) = \int_0^h Q'(x)\,dx = \int_h^{2h} Q'(x)\,dx = Q_1(B)\,.$$

Solution order n

Now suppose we know how to share the volume up to degree n. Maybe we distributed many slices up to a quantity corresponding to a total thickness H.



Introducing the discrepancy Λ between the share of A and the share of B, we have

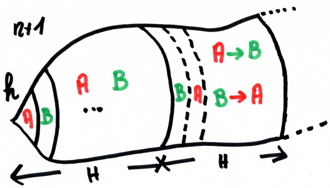
$$\Lambda = \int_A Q'(x)\,dx - \int_B Q'(x)\,dx = 0\,.$$

Solution order $n+1$

If $Q(x) = P_{n+1}(x)$, we have $Q'(x) = c_n x^n + p_{n-1}(x)$. If we apply to case $n+1$ the same slicing strategy as for case n, we will commit a error that will only depend on c_n the coefficient of the term of highest degree.
But this coefficient is invariant under translation, we have for example

$$c_n(x+H)^n = c_n x^n + (\text{polynomial of degree } n-1)\,.$$

Therefore, if we continue after H the same slicing strategy, but attributing to B that was given to A and vice versa, we compensate this way the excess or deficiency generated by the previous cutting. Our problem is solved.



The Prouhet-Thue-Morse sequence

The solution is then: If $Q(x)$ is of degree n, cut 2^n slices of equal thickness h and distribute them between A and B following the sequence:

- $n = 1$, AB;
- $n = 2$, $ABBA$;
- $n = 3$, $ABBABAAB$;
- $n = 4$, $ABBABAABBAABABBA$ etc.

We have generated the celebrated Prouhet-Thue-Morse sequence which possesses many interesting properties and applications.
Surprisingly, even in astrophysics...

Sky background subtraction

Suppose one has to observe, say a galaxy in IR, against a sky background. The usual procedure is to achieve a pose of type A on the galaxy, which includes the sky background, and a second pose of type B on the sky only. That is

$$A = \text{galaxy} + \text{sky}\,, \quad B = \text{sky}\,.$$

If the sky background is constant, we obtain the galaxy by a simple subtraction.

$$\text{galaxy} = A - B\,.$$

Now, if the sky varies with time according to a polynomial of order n, one gets rid of the sky by planning observations according to the Prouhet-Thue-Morse sequence and performing the operation

$$\text{galaxy} = \sum A - \sum B\,.$$

For instance, if $n = 1$, it is better to do $ABBA$ rather than $ABAB$.
If $n = 2$, the sky varies quadratically, the poses should be $ABBABAAB$.

Multigrad multiplets

If $Q(x)$ is of degree n, $Q'(x)$ is of degree $n-1$ and $\frac{d^n}{dx^n}Q'(x) = 0$. One can replace the derivative by its finite difference operator, which is exact up to degree n and will be zero for polynomial of degree $n-1$. If we set $h = 1$ and -1 for A and $+1$ for B, the Prouhet-Thue-Morse sequence is just the sequence of the coefficients of a finite difference operator of degree n.
One can therefore solve some kind of diophantine equations of the form :

$$\sum_{k=1}^{N} p_k^r = \sum_{k=1}^{N} q_k^r, \quad \text{for } r = 1, \ldots, n.$$

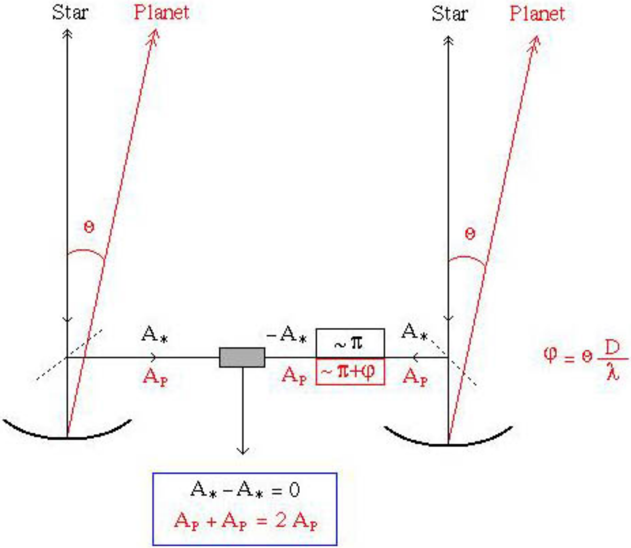
For example, for $n = 2$ and the integer sequence 12345678, we get

$$1+4+6+7 = 2+3+5+8 \quad = 18,$$
$$1^2+4^2+6^2+7^2 = 2^2+3^2+5^2+8^2 \quad = 102.$$

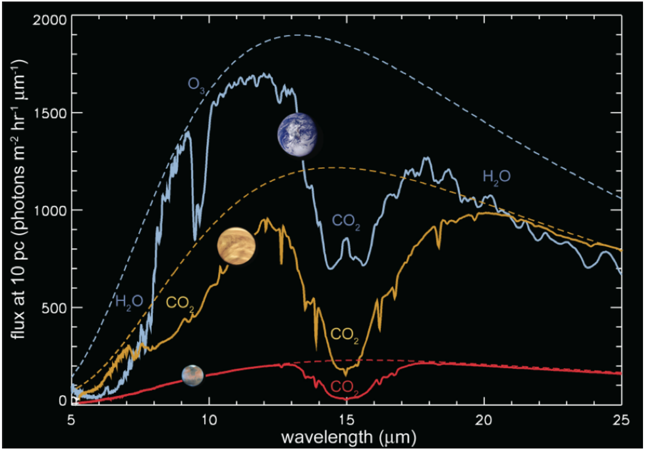
For short $\{1,4,6,7\} \overset{2}{=} \{2,3,5,8\}$.

The Bracewell nulling interferometer

Principle



Bio-tracers are the absorption bands of: O_3, H_2O et CO_2, in the thermal IR on a 6μm-17μm bandwidth.



Doing spectroscopy with the Bracewell nulling interferometer

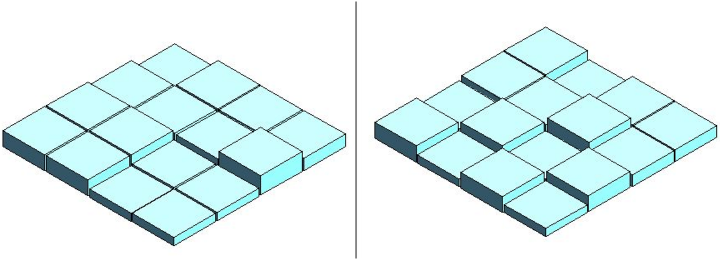
Limitations

- The phase shift of π is obtained by adding an optical path difference (opd) of $\lambda_0/2$ on one arm of the interferometer.
- The light from the star will be cancelled only for the wavelengths that are *odd* multiple of $\lambda_0/2$.
- The interference will be constructive, for the planet, only if the angle of sight θ induce a *supplementary* opd which is an *even* multiple of $\lambda_0/2$.

The device is chromatic. It fails to cancel the starlight within the full useful bandwidth: $6\,\mu$m-$17\,\mu$m.
There remains a residual stellar light whose amplitude depends upon wavelength: *the chromatic function* Λ. film2D31

How to cancel Λ?

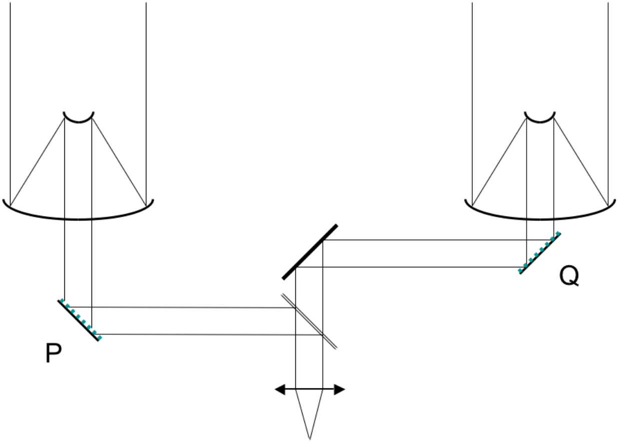
The idea is to divide the wavefront, like the salami, into many sub-pupils by two "chessboards" of cells each assigned to a certain opd.



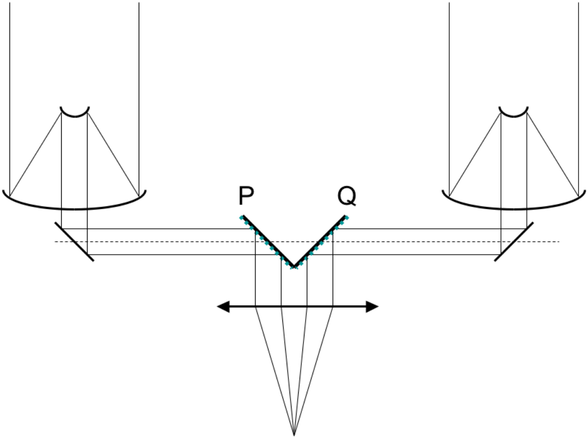
P (even pupil) Q (odd pupil)

opd $= 2k \times \lambda_0/2$ opd $= (2k+1) \times \lambda_0/2$

Michelson setting (uniaxial)



Fizeau setting (multiaxial)



The iterative Bracewell interferometer

The effect of the cells, upon the interferometer PSF, depends on their intrinsic opd's, their positions and their shape. We neglect below their positions (super Michelson) and the shape does not matter very much (sinc).
The Bracewell interferometer is fully destructive for wavelength $\lambda = \lambda_0$. We have the amplitude

$$\Lambda = 1 + (-1) = 0\,.$$

For $\lambda \neq \lambda_0$, we get

$$\Lambda = 1 + (-1)^{\lambda_0/\lambda} \neq 0\,.$$

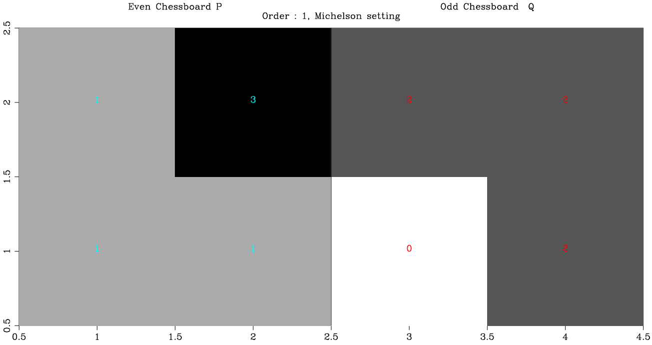
If we set $(-1)^{\lambda_0/\lambda} \equiv e^{i\pi\lambda_0/\lambda} = z$, we get $\Lambda = 1 + z$. That is $\lambda = \lambda_0$ induce a root of order one on Λ. One way to obtain a shallow chromatic function around λ_0 is to have a *multiple* root. That is

$$\Lambda = (1 + z)^n\,.$$

The higher n the deeper the nulling around λ_0.

The achromatic virtues of Pascal's triangle.

This procedure generates opd's in number according to the binomial coefficients. For $n = 3$, we get $(1+z)^3 = 1 + 3z + 3z^2 + z^3$, producing 1 opd of 0, 3 of $\lambda_0/2$, 3 of $2\lambda_0/2$ and 1 of $3\lambda_0/2$. In order to take avantage of the off-axis planet "trick", we arrange the even opd's on one chessboard and the odd ones on the other.



Binomial coefficients integer sequences are multigrad

Indeed Λ *is* a polynomial upon z, but *not* upon λ. However because $z = -1$ is a root of order n, the Taylor expansion of Λ upon λ will be cancelled up to its $(n-1)$-th order. This can also be seen by using the multigrad theorem

$$\text{If} \quad \{p_N\} \stackrel{n}{=} \{q_N\}, \quad \text{then} \quad \{p_N, q_N + c\} \stackrel{n+1}{=} \{q_N, p_N + c\}\,.$$

Starting with $\{0\} \stackrel{0}{=} \{1\}$ and with $c = 1$, we generate a binomial coefficient sequence of integer: $\{0,2\} \stackrel{1}{=} \{1,1\}$, $\{0,2,2,2\} \stackrel{2}{=} \{1,1,1,3\}$, etc.
Indeed

$$0+2+2+2 = 1+1+1+3$$
$$0^2+2^2+2^2+2^2 = 1^2+1^2+1^2+3^2\,.$$

A binomial coefficient sequence of integers is a compact version of the Prouhet-Thue-Morse sequence. It will solve the salami problem if we were given many identical salamis.

Intrinsic values of the iterative Bracewell interferometer opd's

Our interferometer of order $r = 0$ is the original Bracewell one $\{0\} \overset{0}{=} \{1\}$. If $\{p_r\}$ stands for the opd's set, divided by $\lambda_0/2$, on the even pupil and $\{q_r\}$ for the odd ones, one obtains the next order by two applications of the mutigrad theorem (in such a way, the number of cells in each pupils remains a square).

$$\{p_{r+1}\} = \{p_r, q_r + 1, q_r + 1, p_r + 2\},$$

$$\{q_{r+1}\} = \{q_r, p_r + 1, p_r + 1, q_r + 2\},$$

$$\{p_{r+1}\} \overset{2r+2}{=} \{q_{r+1}\}.$$

But we still don't know *where* to place the cells on the even and odd pupils.

Arrangement of the cells on the pupils

Lets call P_r and Q_r the physical arrangement of the phase shifters whose values are given by the two sets $\{p_r\}$ and $\{q_r\}$.

Detailed analytical work shows that in order to preserve, as far as possible, the nulling power of the device, the cells must be placed in such a way that $P_r - Q_r$ is a finite difference differential operator of high order acting on the spatial repartition of the light on the focal plane.

One can achieve this objective with the following iterative arrangement

$$P_{r+1} = \begin{bmatrix} Q_r + 1 & P_n + 2 \\ P_r & Q_r + 1 \end{bmatrix}, \quad Q_{r+1} = \begin{bmatrix} P_r + 1 & Q_n + 2 \\ Q_r & P_r + 1 \end{bmatrix}.$$

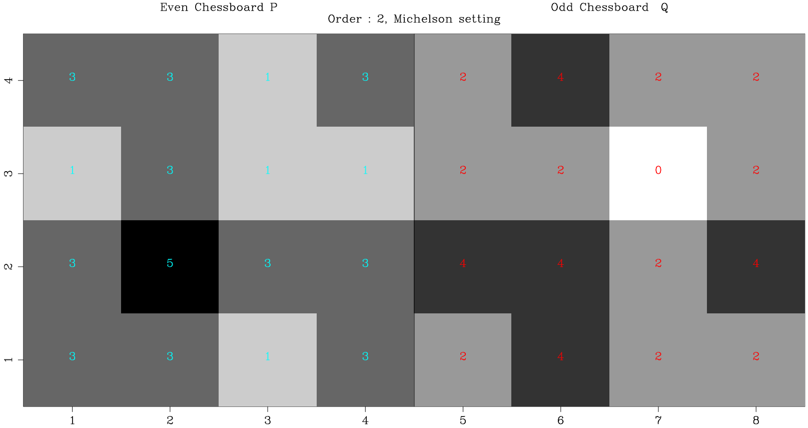
In fact, at first stage we get a gradient, and the process continues on

$$P_1 - Q_1 = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix},$$

$$P_{r+1} - Q_{r+1} = \begin{bmatrix} Q_r - P_r & P_r - Q_r \\ P_r - Q_r & Q_r - P_r \end{bmatrix}.$$

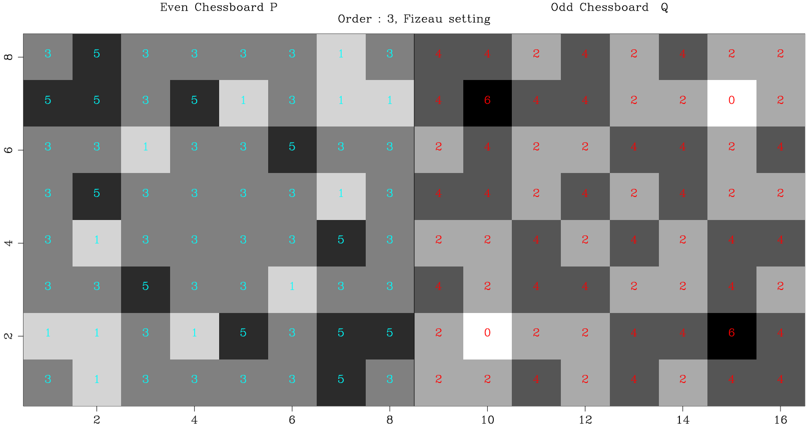
Michelson setting order $r = 2$, i.e. $(1+z)^5$

Even P_2 pupil   Odd Q_2 pupil

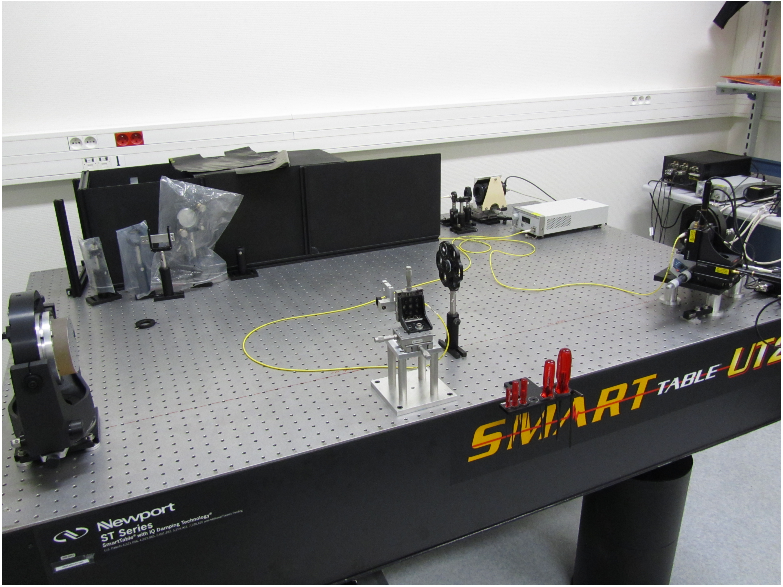


Fizeau (P_r^*, Q_r^*) chessboards, e.g. $r = 3$

The Fizeau chessboard are axisymmetric version of the Michelson ones.
The electric field in the focal plane is canceled, at $\lambda = \lambda_0$, by a single mode optical fiber.



The “Damned” optical bench



Global properties

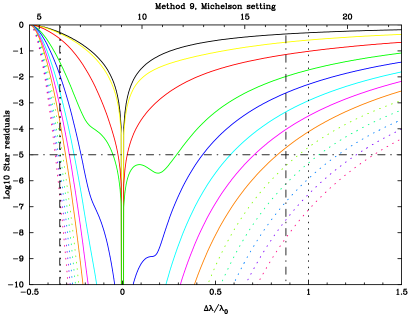
As shown in the table below, Fizeau chessboards retain most optimal properties of Michelson ones.

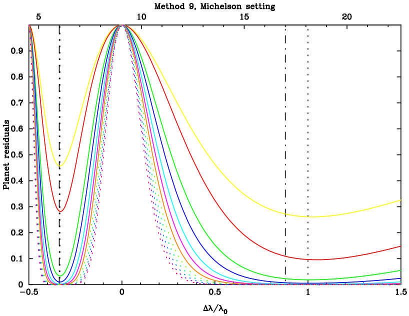
Type	$n_0 \times n_0$	*Chromatic* nul n	*Spatial nul*
co-axial (P_r, Q_r)	$2^r \times 2^r$	$2r$	$r-1$
multi-axial (P_r^*, Q_r^*)	$2^r \times 2^r$	$2r-1$	$r-2$, if r even $r-1$, if r odd

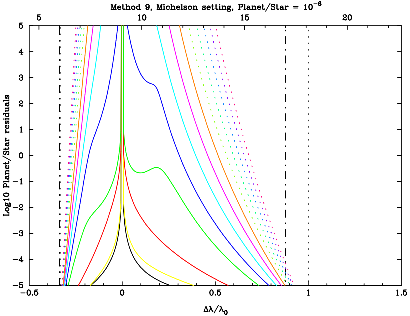
Useful nulling bandwidth : $\frac{2}{3}\lambda_0, 2\lambda_0$, e.g. $\Delta\lambda = (6\mu\text{m}, 18\mu\text{m})$ for $\lambda_0 = 9\mu$m.

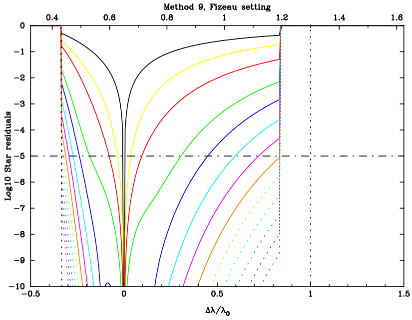
The following graphs show the performances of the devices. The colors indicate the increasing r orders. Order $r = 7$ (orange) meets the Darwin space project specifications.

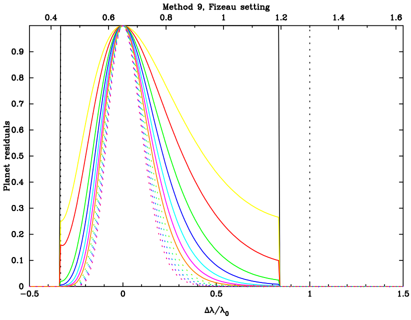
film2D31 film2D32

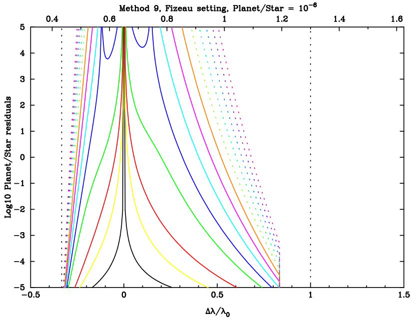







Method 9, Fizeau setting
Log10 Star residuals
Δλ/λ0


Method 9, Fizeau setting
Planet residuals
Δλ/λ0


Method 9, Fizeau setting, Planet/Star = 10^-6
Log10 Planet/Star residuals
Δλ/λ0